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On the Rate of Convergence and the Voronovskaya Theorem for the Poisson Integrals for Hermite and Laguerre Expansions

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The aim of this paper is the study of a rate of convergence of Poisson integrals for Hermite and Laguerre expansions. We prove also the Voronovskaya type theorem and we present boundary value problems related to these integrals.

Key Words: Poisson integral; Hermite and Laguerre expansions; rate of convergence; Voronovskaya theorem; boundary value problems.

1. INTRODUCTION

Muckenhoupt in [4] studied Poisson integrals for Laguerre polynomial expansions. He considered the Poisson integral $A(f)(r,x)$ of a function $f \in L^p(z^\alpha \exp(-z))$, $\alpha > -1$, defined by

$$A(f)(r,x) = A(f;r,x) = \int_0^\infty K(r,x,z)\, f(z)\, z^\alpha \exp(-z)\, dz, \qquad 0<r<1 \tag{1}$$

with the Poisson kernel

$$K(r,x,z) = \sum_{n=0}^{\infty} \frac{r^n n!}{\Gamma(n+\alpha+1)} L_n^\alpha(x)\, L_n^\alpha(z)$$
$$= \frac{(rxz)^{-\frac{\alpha}{2}}}{1-r} \exp\left(\frac{-r(x+z)}{1-r}\right) I_\alpha\left(\frac{2(rxz)^{\frac{1}{2}}}{1-r}\right),$$

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where L_n^α is the nth Laguerre polynomial and I_α is the modified Bessel function [3],

$$I_\alpha(s) = \sum_{n=0}^{\infty} \frac{s^{\alpha+2n}}{2^{\alpha+2n} n!\, \Gamma(\alpha+n+1)}.$$

Muckenhoupt proved in [4] that if $f \in L^p(z^\alpha \exp(-z))$, then

(a) $\|A(f; r, \cdot)\|_p \leqslant \|f(\cdot)\|_p$, $1 \leqslant p \leqslant \infty$,
(b) $\|A(f; r, \cdot) - f(\cdot)\|_p \to 0$ as $r \to 1^-$ for $1 \leqslant p < \infty$,
(c) $\lim_{r \to 1^-} A(f; r, x) = f(x)$ almost everywhere in $[0, \infty)$, $1 \leqslant p \leqslant \infty$.

The symbol $\|f\|_p$ is used here to denote the pth norm of a function f defined on $\mathbb{R}_+ = [0, +\infty)$ with respect to the measure $z^\alpha \exp(-z)\, dz$.

Reference [4] also considered the Poisson integral of a function $f \in L^p(\exp(-z^2))$ for Hermite expansions defined by

$$B(f)(r, x) = B(f; r, x) = \int_{-\infty}^{\infty} P(r, x, z)\, f(z) \exp(-z^2)\, dz, \qquad 0 < r < 1,$$

where

$$P(r, x, z) = \sum_{n=0}^{\infty} \frac{r^n H_n(x)\, H_n(z)}{\sqrt{\pi}\, 2^n n!} = \frac{1}{\sqrt{\pi(1-r^2)}} \exp\left(\frac{-r^2x^2 + 2rxz - r^2z^2}{1-r^2}\right)$$

and H_n is the nth Hermite polynomial. Muckenhoupt obtained the following result: If $f \in L^p(\exp(-z^2))$, then

(a) $\|B(f; r, \cdot)\|_p \leqslant \|f(\cdot)\|_p$, $1 \leqslant p \leqslant \infty$,
(b) $\|B(f; r, \cdot) - f(\cdot)\|_p \to 0$ as $r \to 1^-$ for $1 \leqslant p < \infty$,
(c) $\lim_{r \to 1^-} B(f; r, x) = f(x)$ almost everywhere, $1 \leqslant p \leqslant \infty$,

where $\|f\|_p$ denotes the norm in $L^p(\exp(-z^2))$ of a function f defined on $\mathbb{R}$.

This note contains some estimates of the rate of convergence of the Poisson integrals $A(f)$, $B(f)$. We state these estimates using the moduli of continuity, severally for $A(f)$ and $B(f)$. We believe that the direct calculations in the case of the Hermite polynomials are more useful than those obtained via the formulas connecting the Hermite and Laguerre polynomials. We prove also the Voronovskaya type theorem (see [1]) for operators $A(f)$ and $B(f)$ and we indicate boundary value problems related to these operators.

2. AUXILIARY RESULTS

In this section we shall give some properties of the above operators, which we shall apply to the proofs of the main theorems.

First we prove

Lemma 2.1. *For each $x \in \mathbb{R}_+$ we have*

$$\begin{aligned}
A(1; r, x) &= 1,\\
A(z; r, x) &= (\alpha+1)(1-r)+rx,\\
A(z^2; r, x) &= r^2x^2+(\alpha+2)(r-1)((\alpha+1)(r-1)-2rx),\\
A(z^3; r, x) &= r^3x^3-3(\alpha+3)\, r^3x^2+3(\alpha+3)(\alpha+2)\, r^3x\\
&\quad -(\alpha+3)(\alpha+2)(\alpha+1)\, r^3+3(\alpha+3)\, r^2x^2\\
&\quad -6(\alpha+3)(\alpha+2)\, r^2x+3(\alpha+3)(\alpha+2)(\alpha+1)\, r^2\\
&\quad +3(\alpha+3)(\alpha+2)\, rx-3(\alpha+3)(\alpha+2)(\alpha+1)\, r\\
&\quad +(\alpha+3)(\alpha+2)(\alpha+1),\\
A(z^4; r, x) &= r^4x^4-4(\alpha+4)\, r^4x^3+6(\alpha+4)(\alpha+3)\, r^4x^2\\
&\quad -4(\alpha+4)(\alpha+3)(\alpha+2)\, r^4x+(\alpha+4)(\alpha+3)(\alpha+2)(\alpha+1)\, r^4\\
&\quad +4(\alpha+4)\, r^3x^3-12(\alpha+4)(\alpha+3)\, r^3x^2\\
&\quad +12(\alpha+4)(\alpha+3)(\alpha+2)\, r^3x-4(\alpha+4)(\alpha+3)(\alpha+2)(\alpha+1)\, r^3\\
&\quad +6(\alpha+4)(\alpha+3)\, r^2x^2-12(\alpha+4)(\alpha+3)(\alpha+2)\, r^2x\\
&\quad +6(\alpha+4)(\alpha+3)(\alpha+2)(\alpha+1)\, r^2+(\alpha+4)(\alpha+3)(\alpha+2)(\alpha+1)\\
&\quad +4(\alpha+4)(\alpha+3)(\alpha+2)\, rx-4(\alpha+4)(\alpha+3)(\alpha+2)(\alpha+1)\, r.
\end{aligned}$$

Proof. Using the definition of L_n^α we get

$$z^n = \sum_{k=0}^{n} \frac{\Gamma(n+\alpha+1)}{\Gamma(k+\alpha+1)} \cdot \frac{n!}{(n-k)!} (-1)^k L_k^\alpha(z).$$

Since [4]

$$\int_0^\infty K(r, x, z)\, L_n^\alpha(z)\, z^\alpha \exp(-z)\, dz = r^n L_n^\alpha(x)$$

and

$$\int_0^\infty K(r, x, z)\, z^\alpha \exp(-z)\, dz = 1, \tag{2}$$

we obtain, from (1) and by elementary calculations, the assertion of Lemma 2.1.

Analogously, by the definition of H_n we have

$$z^{2n}=\sum_{k=0}^{n}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(k+\frac{1}{2})}\binom{n}{k}\frac{1}{2^{2k}}H_{2k}(z)$$

and

$$z^{2n+1}=\sum_{k=0}^{n}\frac{\Gamma(n+\frac{3}{2})}{\Gamma(k+\frac{3}{2})}\binom{n}{k}\frac{1}{2^{2k+1}}H_{2k+1}(z).$$

Using [4]

$$\int_{-\infty}^{\infty}P(r,x,z)\,H_n(z)\exp(-z^2)\,dz=r^nH_n(x),$$

$$\int_{-\infty}^{\infty}P(r,x,z)\exp(-z^2)\,dz=1,$$

we can prove

LEMMA 2.2. *For each $x\in\mathbb{R}$ we have*

$$B(1;r,x)=1,$$
$$B(z;r,x)=rx,$$
$$B(z^2;r,x)=r^2x^2+\tfrac{1}{2}(1-r^2),$$
$$B(z^3;r,x)=-\tfrac{3}{2}r^3x+r^3x^3+\tfrac{3}{2}rx,$$
$$B(z^4;r,x)=\tfrac{3}{4}r^4-3r^4x^2+r^4x^4-\tfrac{3}{2}r^2+3r^2x^2+\tfrac{3}{4}.$$

Applying Lemma 2.1 and Lemma 2.2 it is easy to prove the following two lemmas.

LEMMA 2.3. *For each $x\in\mathbb{R}_+$ the following equalities hold*

$$A(z-x;r,x)=(1-r)(1+\alpha-x),$$

$$\begin{aligned}A((z-x)^2;r,x)=(1-r)\{&x^2(1-r)+2(\alpha+2)\,rx-2(\alpha+1)\,x\\&+(\alpha+2)(\alpha+1)(1-r)\},\end{aligned}$$

$$\begin{aligned}A((z-x)^4;r,x)=(1-r)^2\,\{&x^4(r-1)^2-4\alpha(r-1)^2\,x^3-4(4r^2-5r+1)\\&+6(\alpha+4)(\alpha+3)\,r^2x^2-12(\alpha+3)(\alpha+2)\,rx^2\\&+6(\alpha+2)(\alpha+1)\,x^2-12(\alpha+3)(\alpha+2)(r-1)\,x\\&+(\alpha+4)(\alpha+3)(\alpha+2)(\alpha+1)(r-1)^2\}.\end{aligned}$$

LEMMA 2.4. *For each* $x \in \mathbb{R}$

$$B(z-x; r, x) = -x(1-r),$$
$$B((z-x)^2; r, x) = (1-r)\{x^2(1-r)+\tfrac{1}{2}(r+1)\},$$
$$B((z-x)^4; r, x) = (1-r)^2\{(r-1)^2 x^4 - 3(r^2-1)x^2 + \tfrac{3}{4}(r+1)^2\}$$

holds.

The above lemmas yield immediately Lemma 2.5 and Lemma 2.6.

LEMMA 2.5. *For every fixed* $x \in \mathbb{R}_+$ *we have*

$$\lim_{r\to 1^-} \frac{1}{1-r} A(z-x, r, x) = 1+\alpha-x,$$

$$\lim_{r\to 1^-} \frac{1}{1-r} A((z-x)^2; r, x) = 2x,$$

$$\lim_{r\to 1^-} \frac{1}{(1-r)^2} A((z-x)^4; r, x) = 12x^2.$$

LEMMA 2.6. *For every fixed* $x \in \mathbb{R}$ *it holds*

$$\lim_{r\to 1^-} \frac{1}{1-r} B(z-x; r, x) = -x,$$

$$\lim_{r\to 1^-} \frac{1}{1-r} B((z-x)^2; r, x) = 1,$$

$$\lim_{r\to 1^-} \frac{1}{(1-r)^2} B((z-x)^4; r, x) = 3.$$

3. RATE OF CONVERGENCE

In this part we shall state some estimates of the rate of convergence of the integrals $A(f)$ and $B(f)$. We shall use the classical modulus of continuity defined by

$$\omega(f, \delta) = \sup_{\substack{0\leqslant t\leqslant \delta\\ x\in Q}} |f(x+t)-f(x)|,$$

where $Q = \mathbb{R}_+$ or $Q = \mathbb{R}$, respectively.

We shall apply the method used in [1, 2].

THEOREM 3.7. *Let* $f \in C(\mathbb{R}_+) \cap L^p(z^\alpha \exp(-z))$. *Then*

$$|A(f; r, x) - f(x)| \leqslant 3\omega(f, \sqrt{(1-r)((x^2+\alpha^2+3\alpha+2)(1-r)+2x((\alpha+2)\, r-\alpha-1))})$$

for $0 < r < 1$ *and* $x \geqslant 0$.

Proof. First we suppose that f is continuously differentiable on $[0, +\infty)$. We have

$$f(z) = f(x) + \int_x^z f'(\tau)\, d\tau.$$

Hence, from (2), Lemma 2.3, and from the Hölder inequality, we obtain

$$\begin{aligned}
|A(f; r, x) - f(x)| &= \left| \int_0^\infty K(r, x, z)\, z^\alpha \exp(-z)(f(z) - f(x))\, dz \right| \\
&\leqslant \int_0^\infty K(r, x, z)\, z^\alpha \exp(-z)\, |f(z) - f(x)|\, dz \\
&\leqslant \sup_{[0, +\infty)} |f'(z)| \int_0^\infty K(r, x, z)\, z^\alpha \exp(-z)\, |z - x|\, dz \\
&\leqslant \sup_{[0, +\infty)} |f'(z)|\, (A(\varphi; r, x))^{\frac{1}{2}} (A(1; r, x))^{\frac{1}{2}} \\
&= \sup_{[0, +\infty)} |f'(z)| \\
&\quad \times \sqrt{(1-r)((x^2+\alpha^2+3\alpha+2)(1-r)+2x((\alpha+2)\, r-\alpha-1))}
\end{aligned}$$

for $0 < r < 1$, $x \geqslant 0$, where $\varphi(z) = (z - x)^2$.

Let $f \in C(\mathbb{R}_+) \cap L^p(z^\alpha \exp(-z))$. We have

$$f(x) - f_\delta(x) = \frac{1}{\delta} \int_0^\delta (f(x) - f(x+\tau))\, d\tau,$$

$$f'_\delta(x) = \frac{1}{\delta} [f(x+\delta) - f(x)],$$

where

$$f_\delta(x) = \frac{1}{\delta} \int_0^\delta f(x+\tau)\, d\tau, \qquad \delta > 0, \quad x \geqslant 0.$$

This implies that f_δ is continuously differentiable on $[0, +\infty)$. Moreover

$$\sup_{[0,+\infty)} |f(x)-f_\delta(x)| \leqslant \omega(f,\delta), \qquad \sup_{[0,+\infty)} |f'_\delta(z)| \leqslant \delta^{-1}\omega(f,\delta). \tag{3}$$

Observe that

$$|A(f;r,x)-f(x)| \leqslant |A(f-f_\delta;r,x)|+|A(f_\delta;r,x)-f_\delta(x)|+|f_\delta(x)-f(x)|.$$

From (3) and the first part of this proof we get

$$\begin{aligned}|A(f_\delta;r,x)-f_\delta(x)|\\ &\leqslant \delta^{-1}\omega(f,\delta)\\ &\quad\times\sqrt{(1-r)((x^2+\alpha^2+3\alpha+2)(1-r)+2x((\alpha+2)\,r-\alpha-1))}.\end{aligned}$$

By (3) we have

$$|A(f-f_\delta;r,x)| \leqslant A(|f-f_\delta|;r,x) \leqslant \sup_{[0,+\infty)} |f(z)-f_\delta(z)| \leqslant \omega(f,\delta),$$

$$|f_\delta(x)-f(x)| \leqslant \omega(f,\delta).$$

Hence

$$\begin{aligned}|A(f;r,x)-f(x)|\\ &\leqslant 2\omega(f,\delta)+\frac{1}{\delta}\,\omega(f,\delta)\\ &\quad\times\sqrt{(1-r)((x^2+\alpha^2+3\alpha+2)(1-r)+2x((\alpha+2)\,r-\alpha-1))}\end{aligned}$$

for $0<r<1$, $x \geqslant 0$ and $\delta>0$.

Setting $\delta=\sqrt{(1-r)((x^2+\alpha^2+3\alpha+2)(1-r)+2x((\alpha+2)\,r-\alpha-1))}$ we obtain the assertion of Theorem 3.7.

Similarly we can prove the following theorem for the operator $B(f)$.

THEOREM 3.8. *Let* $f \in C(\mathbb{R}) \cap L^p(\exp(-z^2))$. *Then*

$$|B(f;r,x)-f(x)| \leqslant 3\omega\left(f, \sqrt{(1-r)(\tfrac{1}{2}\,(r+1)+x^2(1-r))}\right)$$

for $0<r<1$ *and* $x \in \mathbb{R}$.

4. THE VORONOVSKAYA TYPE THEOREMS

THEOREM 4.9. *Let $x \in [0, +\infty)$. If $f \in C(\mathbb{R}_+) \cap L^p(z^\alpha \exp(-z))$, f is of the class C^1 in a certain neighbourhood of a point x, and $f''(x)$ exists, then*

$$\lim_{r \to 1^-} \frac{1}{1-r} [A(f; r, x) - f(x)] = (1+\alpha-x) f'(x) + x f''(x). \tag{4}$$

Proof. Let

$$\psi(z, x) = \begin{cases} \dfrac{f(z)-f(x)-(z-x) f'(x)-\frac{1}{2} f''(x)(z-x)^2}{(z-x)^2} & \text{for} \quad z \neq x, \\ 0 & \text{for} \quad z = x. \end{cases} \tag{5}$$

By assumption we have

$$\lim_{z \to x} \psi(z, x) = 0$$

and the function ψ is continuous on $[0, +\infty)$ as a function of z only.

From (1), (5), and Lemma 2.1 we get

$$\begin{aligned} A(f(z); r, x) - f(x)) = f'(x) A(z-x; r, x) + \tfrac{1}{2} f''(x) A((z-x)^2; r, x) \\ + A(\psi(z, x)(z-x)^2; r, x). \end{aligned} \tag{6}$$

Using the Hölder inequality we obtain

$$|A(\psi(z, x)(z-x)^2; r, x)| \leqslant (A(\psi^2(z, x); r, x))^{\frac{1}{2}} (A((z-x)^4; r, x))^{\frac{1}{2}}. \tag{7}$$

Moreover, the function $\varphi(z, x) := \psi^2(z, x)$, $z \geqslant 0$ is such that

$$\lim_{z \to x} \varphi(z, x) = 0$$

and $\varphi \in L^s(z^\alpha \exp(-z))$ for some $s \geqslant 1$. From this and in view of Theorem 3.7 we get

$$\lim_{r \to 1^-} A(\psi^2(z, x); r, x) \equiv \lim_{r \to 1^-} A(\varphi(z, x); r, x) = \varphi(x, x) = 0. \tag{8}$$

Using (8) and Lemma 2.5 we obtain from (7)

$$\lim_{r \to 1^-} \frac{1}{1-r} A(\psi(z, x)(z-x)^2; r, x) = 0. \tag{9}$$

Now, from (6), (9), and Lemma 2.5 we get

$$\lim_{r\to 1^-} \frac{1}{1-r}\,[A(f;r,x)-f(x)]=(1+\alpha-x)\,f'(x)+xf''(x).$$

This ends the proof of (4).

In a similar fashion, using Theorem 3.8 we can prove the following Voronovskaya theorem for operator $B(f)$.

THEOREM 4.10. *Let $x\in\mathbb{R}$. If $f\in C(\mathbb{R})\cap L^p(\exp(-z^2))$, f is of the class C^1 in a certain neighbourhood of a point x, and $f''(x)$ exists, then*

$$\lim_{r\to 1^-} \frac{1}{1-r}\,[B(f;r,x)-f(x)]=-xf'(x)+\frac{1}{2}\,f''(x).$$

5. BOUNDARY VALUE PROBLEMS

In this section we indicate boundary value problems related to the operators $A(f)$ and $B(f)$.

THEOREM 5.11. *If $f\in L^p(z^\alpha\exp(-z))$, then $A(f)$ is of the class C^∞ in the set $D=\{(r,x)\colon 0<r<1,\ \ x\geqslant 0\}$ and $A(f)$ is a solution of the heat-diffusion equation*

$$r\,\frac{\partial u(r,x)}{\partial r}+(1+\alpha-x)\,\frac{\partial u(r,x)}{\partial x}+x\,\frac{\partial^2 u(r,x)}{\partial x^2}=0$$

in D.

Proof. Let $0<\rho_1<\rho_2<1$ and $0\leqslant C_1<C_2$. Consider the set

$$D(\rho_1,\rho_2,C_1,C_2)=\{(r,x)\colon \rho_1<r<\rho_2,\ C_1<x<C_2\}.$$

Let $n,m\in\mathbb{N}$. Applying the induction and the following formula [3, p. 30]

$$\frac{d}{dz}\,(z^{-\nu}I_\nu(z))=z^{-\nu}I_{\nu+1}(z)$$

we note that the integral

$$\int_0^\infty \frac{\partial^{n+m}}{\partial r^n\partial x^m}\,K(r,x,z)\,f(z)\,z^\alpha\exp(-z)\,dz \qquad (10)$$

is a linear combination of integrals of the form

$$\int_0^\infty z^\beta r^\gamma x^\mu (1-r)^\nu \exp\left(-\frac{r(x+z)}{1-r}\right)\left(\frac{2\sqrt{rxz}}{1-r}\right)^{-\tau} \times I_\tau\left(\frac{2\sqrt{rxz}}{1-r}\right) f(z)\, z^\alpha \exp(-z)\, dz,$$

where $\tau \geqslant \alpha$, $\beta \geqslant 0$, γ, μ, $\nu \in \mathbb{R}$.

By induction we have

$$2^{2k} k!\, \Gamma(\nu+k+1) \geqslant \Gamma(\nu+1)(2k)!$$

for $k \in \mathbb{N}$ and $\nu \geqslant -\frac{1}{2}$. Hence

$$|I_\nu(z)| \leqslant \frac{1}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^\nu \exp z$$

for $\nu \geqslant -\frac{1}{2}$ and $z \geqslant 0$. Thus, by the formula [3, p. 29]

$$I_\tau(z) = \frac{2(\tau+1)}{z} I_{\tau+1}(z) + I_{\tau+2}(z)$$

we have

$$|I_\tau(z)| \leqslant K_1 \left(\frac{z}{2}\right)^\tau \exp 2z$$

for every $\tau \geqslant \alpha > -1$ and $z \geqslant 0$, where K_1 is a positive constant.

Hence

$$\left|\int_0^\infty z^\beta r^\gamma x^\mu (1-r)^\nu \exp\left(-\frac{r(x+z)}{1-r}\right)\left(\frac{2\sqrt{rxz}}{1-r}\right)^{-\tau} \times I_\tau\left(\frac{2\sqrt{rxz}}{1-r}\right) f(z)\, z^\alpha \exp(-z)\, dz\right| \leqslant K_2 \int_0^\infty \exp(-az + b\sqrt{z}\,)\, z^\beta\, |f(z)|\, z^\alpha \exp(-z)\, dz$$

for $(r, x) \in D(\rho_1, \rho_2, C_1, C_2)$, where a, b, and K_2 are positive constants.

By the Hölder inequality we get

$$\int_0^\infty z^\beta \exp(-az + b\sqrt{z})\, |f(z)|\, z^\alpha \exp(-z)\, dz \leqslant \left(\int_0^\infty z^{q\beta+\alpha} \exp(-aqz + bq\sqrt{z} - z)\, dz\right)^{\frac{1}{q}} \|f\|_p,$$

where $\frac{1}{p}+\frac{1}{q}=1$. This implies that

$$\left|\int_0^\infty z^\beta r^\gamma x^\mu (1-r)^\nu \exp\left(-\frac{r(x+z)}{1-r}\right)\left(\frac{2\sqrt{rxz}}{1-r}\right)^{-\tau} \right.$$
$$\left. \times I_\tau\left(\frac{2\sqrt{rxz}}{1-r}\right) f(z)\, z^\alpha \exp(-z)\, dz\right|$$
$$\leqslant K_2 \left(\int_0^\infty z^{q\beta+\alpha} \exp(-aqz+bq\sqrt{z}-z)\, dz\right)^{\frac{1}{q}} \|f\|_p$$

for $(r, x) \in D(\rho_1, \rho_2, C_1, C_2)$.

Observe that the integral

$$\int_0^\infty z^{q\beta+\alpha} \exp(-aqz+bq\sqrt{z}-z)\, dz$$

is convergent, hence the integral (10) is uniformly convergent on $D(\rho_1, \rho_2, C_1, C_2)$. This implies that

$$\frac{\partial^{n+m}}{\partial r^n \partial x^m} \int_0^\infty K(r, x, z)\, f(z)\, z^\alpha \exp(-z)\, dz$$
$$= \int_0^\infty \frac{\partial^{n+m}}{\partial r^n \partial x^m} K(r, x, z)\, f(z)\, z^\alpha \exp(-z)\, dz.$$

Consequently, the function $A(f)$ is of class C^∞ in D.

It is easy to verify that

$$r\,\frac{\partial K(r, x, z)}{\partial r} + (1+\alpha-x)\,\frac{\partial K(r, x, z)}{\partial x} + x\,\frac{\partial^2 K(r, x, z)}{\partial x^2} = 0$$

in D for every $z \in (0, +\infty)$. This completes the proof of Theorem 5.11.

Corollary 5.12. *If $f \in L^p(z^\alpha \exp(-z))$, then the function $A(f)$ is a solution of the problem*

$$r\,\frac{\partial u(r, x)}{\partial r} + (1+\alpha-x)\,\frac{\partial u(r, x)}{\partial x} + x\,\frac{\partial^2 u(r, x)}{\partial x^2} = 0 \qquad \text{in } D,$$

$$\lim_{r\to 1^-} u(r, x) = f(x) \qquad \text{almost everywhere in } [0, +\infty).$$

COROLLARY 5.13. *If* $f \in C(\mathbb{R}_+) \cap L^p(z^\alpha \exp(-z))$, *then the function* $A(f)$ *is a solution of the problem*

$$r\frac{\partial u(r,x)}{\partial r} + (1+\alpha-x)\frac{\partial u(r,x)}{\partial x} + x\frac{\partial^2 u(r,x)}{\partial x^2} = 0 \qquad \text{in } D,$$

$$\lim_{r\to 1^-} u(r,x) = f(x), \qquad x \in [0, +\infty).$$

For $B(f)$ we obtain similar results:

THEOREM 5.14. *If* $f \in L^p(\exp(-z)^2)$, *then* $B(f)$ *is of the class* C^∞ *in the set* $D = \{(r,x) : 0 < r < 1, \ x \in \mathbb{R}\}$ *and* $B(f)$ *is a solution of the equation*

$$2r\frac{\partial u(r,x)}{\partial r} - 2x\frac{\partial u(r,x)}{\partial x} + \frac{\partial^2 u(r,x)}{\partial x^2} = 0$$

in D.

COROLLARY 5.15. *If* $f \in L^p(\exp(-z^2))$, *then the function* $B(f)$ *is a solution of the problem*

$$2r\frac{\partial u(r,x)}{\partial r} - 2x\frac{\partial u(r,x)}{\partial x} + \frac{\partial^2 u(r,x)}{\partial x^2} = 0 \qquad \text{in } D,$$

$$\lim_{r\to 1^-} u(r,x) = f(x) \qquad \text{almost everywhere.}$$

COROLLARY 5.16. *If* $f \in C(\mathbb{R}) \cap L^p(\exp(-z)^2)$, *then the function* $B(f)$ *is a solution of the problem*

$$2r\frac{\partial u(r,x)}{\partial r} - 2x\frac{\partial u(r,x)}{\partial x} + \frac{\partial^2 u(r,x)}{\partial x^2} = 0 \qquad \text{in } D,$$

$$\lim_{r\to 1^-} u(r,x) = f(x), \qquad x \in \mathbb{R}.$$

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